Generalized synchronization in fractional order systems

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The generalized synchronization of fractional order systems is investigated, including the synchronization between two different fractional order systems with the same order, the synchronization between two different fractional order systems with no orders the same, and the synchronization between a classical system and its corresponding fractional order system with mismatched parameters. The mechanism for the occurrence of generalized synchronization of fractional order systems is clarified, the necessary and sufficient conditions are given, and several methods to detect generalized synchronization are discussed. The relationship between generalized synchronization and the equivalence of fractional order systems is also considered.

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I. INTRODUCTION

In 1695, Leibniz wrote a letter to L'Hôpital asking whether or not the meaning of derivatives with integer orders could be naturally generalized to derivatives with noninteger orders. L'Hôpital felt somewhat curious about this question and then asked a simple question as reply: "What if the order will be 1/2?" In a following letter on September 30 of the same year, Leibniz anticipated: "It will lead to a paradox, from which one day useful consequences will be drawn." This special date, September 30, 1695, is then regarded as the exact birthday of the fractional calculus [1–3]. Although fractional calculus is a 300-year-old mathematical topic, it did not attract enough attention until recent decades. Nowadays, fulfilling the expectation of those famous scientists, the applications of fractional calculus to physics and engineering arouse more and more interest, as in acoustic and thermal systems, rheology and mechanical systems, signal processing and systems identification, control and robotics, etc. [1–4]. Moreover, many systems modeled with the help of fractional calculus display rich fractional order dynamics, such as viscoelastic systems [5], colored noise [6], boundary layer effects in ducts [7], electromagnetic waves [8], fractional kinetics [9,10], and electrode-electrolyte polarization [11,12]. As the interdisciplinary applications are described elegantly with the help of fractional calculus, many authors began to investigate the chaotic dynamics of fractional order nonlinear dynamical systems [13–16]. According to the Poincaré-Bendixson theorem [17], chaos cannot occur in two-dimensional autonomous ordinary differential equations; in other words, if chaos appears in autonomous ordinary differential equations then its dimension is at least three in the classical sense. However, with the introduction of fractional derivatives, it was proved that chaotic behavior can exist in fractional order systems with order less than three; for example, see Ref. [13] for the chaotic fractional order Chua system of order 2.7, Ref. [14] for chaotic behaviors of the fractional order "jerk" model with order as low as 2.1, Ref. [15] for the chaotic fractional order Lorenz system, and Ref. [18] for the chaotic fractional order Chen and Lü system, etc.

Historically, the analysis of synchronization phenomena in the evolution of dynamical systems started in the 17th century with the finding of Huygens that two weakly coupled pendulum clocks (hanging from the same beam) become synchronized in phase [19], and the search for chaos synchronization has become a hot topic since 1990 [20]. Usually two dynamical systems are said to be synchronized if the distance between their states converges to zero when the time t tends to infinity. Now this kind of synchronization is called complete synchronization (CS), and there exist several other extended concepts of synchronization, namely, phase synchronization, lag synchronization, and generalized synchronization (GS), etc. [19,20]. Of these, GS is of significant practical importance because in practice we can never construct two absolutely identical circuits, and it is crucial that GS can reveal the essential relationship between the drive and response systems [21–30]. More recently, synchronization of chaotic fractional order systems has become a new focus of interest [16,18]. In Ref. [16], Li *et al.* first numerically realized the CS of chaotic fractional order systems and then Deng and Li further systematically investigated this topic in Ref. [18]. To the best of our knowledge, there are no reports on the other kinds of synchronization of fractional order systems to this day. It is well known that synchronization essentially is a conditional stability issue. For nonlinear fractional differential systems, determining stability is a difficult task although there has already been some progresses in linear fractional differential systems (with or without delays) and nonlinear ordinary differential systems (with or without delays) [31]. In the present paper, the GS of fractional order systems is investigated, including the synchronization between two different fractional order systems with the same orders, the synchronization between two different fractional order systems with different orders, and the synchronization between a classical system and its corresponding fractional order system with mismatched parameters. The mechanism and the necessary and sufficient conditions for the occurrence of GS of fractional order systems are clarified and presented, and several methods to detect GS of fractional order systems are discussed. The relationship between the GS and the equivalence of fractional order systems is also considered.

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In what follows, the definition of fractional derivative is introduced briefly. There are several definitions for the fractional differential operator. Hereafter the following definition is used:

$$D_*^{\alpha}y(t) = J^{m-\alpha}y^{(m)}(t), \quad \alpha > 0,$$

where $m=\lceil\alpha\rceil$, i.e., m is the first integer that is not less than α, $y^{(m)}$ is the general m-order derivative, and J^{β} is the β-order Riemann-Liouville integral operator, which is expressed as follows:

$$J^{\beta}z(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}z(\tau)d\tau.$$

The operator D_*^{α} is generally called the "α-order Caputo differential operator" [32]. Clearly, the Caputo differential operator is a nonlocal operator, like other defined fractional derivatives, so it will induce some difficulties in doing not only numerical computation but also theoretical analysis.

II. THEORETICAL ANALYSIS FOR THE GENERALIZED SYNCHRONIZATION OF FRACTIONAL ORDER SYSTEMS

First, the definition of GS is generalized from classical systems to fractional order systems. Consider the following unidirectionally coupled fractional order systems (systems with skew product structure) [21]:

$$D_*^{\bar{\alpha}}\mathbf{x} = \mathbf{f}(\mathbf{x}),$$

$$D_*^{\bar{\beta}}\mathbf{y} = \mathbf{g}(\mathbf{y},\mathbf{u}) = \mathbf{g}(\mathbf{y},\mathbf{h}(\mathbf{x})), \quad (1)$$

where $\mathbf{x}=(x_1,x_2,\ldots,x_n)^T \in \mathbb{R}^n$, $\mathbf{y}=(y_1,y_2,\ldots,y_m)^T \in \mathbb{R}^m$, $D_*^{\bar{\alpha}}\mathbf{x}=(D_*^{\alpha_1}x_1,D_*^{\alpha_2}x_2,\ldots,D_*^{\alpha_n}x_n)^T$, $D_*^{\bar{\beta}}\mathbf{y}=(D_*^{\beta_1}y_1,D_*^{\beta_2}y_2,\ldots,D_*^{\beta_m}y_m)^T$, $\alpha_i,\beta_j \in \mathbb{R}^+$, $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^n$, $\mathbf{g}:\mathbb{R}^m\to\mathbb{R}^m$, and $\mathbf{u}(t)=(u_1(t),u_2(t),\ldots,u_k(t))$ with $u_j(t)=h_j(\mathbf{x}(t,\mathbf{x}_0))$. The first and second systems in (1) are called the drive and response system, respectively; $\mathbf{u}=\mathbf{h}(\mathbf{x})$ is the driving signal. It is said that (1) possesses the property of GS if the following holds: There exists a transformation $\mathbf{H}:\mathbb{R}^n\to\mathbb{R}^m$, a manifold $M=\{(\mathbf{x},\mathbf{y}):\mathbf{y}=\mathbf{H}(\mathbf{x})\}$, and a subset $B=B_x\times B_y\subset\mathbb{R}^n\times\mathbb{R}^m$ with $M\subset B$ such that all trajectories of (1) with initial conditions in the basin B approach M as time t goes to infinity. Essentially it is a rather weak condition on the stability of the response to the behavior of the driving system.

A. Necessary and sufficient conditions for the occurrence of generalized synchronization of fractional order systems

It is well known that Lyapunov exponents (LEs) describe the attributes of the time series of dynamical systems, so for the GS of fractional order systems LEs also play an important role. The dynamical behavior of the unidirectionally coupled systems (1) is characterized by the Lyapunov exponent spectrum $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_{n+m}$. Since the first system in (1) is independent, the dynamical behavior of the second system in (1) relies on the first one and its LEs are called conditional Lyapunov exponents (CLEs), the Lyapunov exponent spectrum of (1) can be classified as LEs of the first system (drive system) $\lambda_1^d\geqslant\cdots\geqslant\lambda_n^d$ and CLEs of the second system (response system) $\lambda_1^r\geqslant\cdots\geqslant\lambda_m^r$. The condition for the occurrence of GS is $\lambda_1^r<0$, which is universal for all systems (fractional order systems, classical systems, time-delayed systems, etc.), because the negative CLEs determine the conditional asymptotical stability of the response system. For the linear fractional order system $D_*^{\alpha}\mathbf{x}=A\mathbf{x}$, where $\alpha\in(0,1)$, $\mathbf{x}\in\mathbb{R}^n$, and $A\in\mathbb{R}^n\times\mathbb{R}^n$, the asymptotical stability condition is $|\arg[\mathrm{spec}(A)]|>\alpha\pi/2$ [33]. This naturally generalizes the result of the classical case $\alpha=1$; obviously it is a more easily satisfied condition compared to the stability condition of a classical system, so we can reasonably predict that GS can be more easily realized in (1) than in its corresponding classical systems. In order to explain the mechanism of the onset of GS more clearly, let us consider a special case of (1) [25]:

$$D_*^{\bar{\alpha}}\mathbf{x} = \mathbf{h}(\mathbf{x},\mathbf{p}_1),$$

$$D_*^{\bar{\beta}}\mathbf{y} = \mathbf{h}(\mathbf{y},\mathbf{p}_2) + \varepsilon\mathbf{B}(\mathbf{x}-\mathbf{y}) = [\mathbf{h}(\mathbf{y},\mathbf{p}_2) - \varepsilon\mathbf{B}\mathbf{y}] + \varepsilon\mathbf{B}\mathbf{x}, \quad (2)$$

where $\mathbf{B}=\{\delta_{ij}\}$ is the coupling matrix, $\delta_{ij}=0$ or 1, and $\delta_{ij}=0$ for $i\neq j$. With increase of the parameter ε, GS may appear in (2). This may be considered as a result of two cooperative processes taking place simultaneously. Since $-\varepsilon\mathbf{B}\mathbf{y}$ brings dissipation into the second system of (2), when ε increases the dissipation grows and at the same time $\varepsilon\mathbf{B}\mathbf{x}$ causes increase of the amplitude of the external signal. With the increase of ε, the dynamical behavior of

$$D_*^{\bar{\beta}}\mathbf{y} = \mathbf{h}(\mathbf{y},\mathbf{p}_2) - \varepsilon\mathbf{B}\mathbf{y} \quad (3)$$

will change from a chaotic oscillation to a periodic one, or even to a stationary state, i.e., the LEs of (3) will become nonpositive or even negative. But it must be noted that this does not mean that GS occurs in (2), because the LEs of (3) differ from the CLEs of (2). The external signal $\varepsilon\mathbf{B}\mathbf{x}$ imposes the dynamics of the first system of (2) on the response system and complicates its dynamical behavior; obviously GS may occur only if the proper dynamics of the drive system is suppressed by dissipation. To more accurately characterize the conditions for the occurrence of GS of the general form (1), the following theorem is presented.

Theorem. GS occurs in (1) if and only if for all $(\mathbf{x}_0,\mathbf{y}_0)\in B$ the response system $D_*^{\bar{\beta}}\mathbf{y}=\mathbf{g}(\mathbf{y},\mathbf{u})=\mathbf{g}(\mathbf{y},\mathbf{h}(\mathbf{x}))$ is asymptotically stable, i.e., $\forall\ \mathbf{y}_{10},\ \mathbf{y}_{20}\in B_y$, $\lim_{t\to+\infty}\|\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_{10})-\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_{20})\|=0$, where $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ are continuous functions in (1).

Proof. According to the existence and uniqueness theorem of fractional differential equations [34], under the conditions of this theorem, (1) has a unique solution for any given initial value.

"$\Rightarrow$" (*necessity*). Because of the occurrence of GS in (1), for any given $(\mathbf{x}_0,\mathbf{y}_{10})\in B$ and $(\mathbf{x}_0,\mathbf{y}_{20})\in B$ there exists a transform function $\mathbf{H}$ such that $\lim_{t\to+\infty}\|\mathbf{y}(t,\mathbf{x}_{10},\mathbf{y}_{10})-\mathbf{H}(\mathbf{x}(t,\mathbf{x}_{10}))\|=0$ and $\lim_{t\to+\infty}\|\mathbf{y}(t,\mathbf{x}_{10},\mathbf{y}_{20})-\mathbf{H}(\mathbf{x}(t,\mathbf{x}_{10}))\|=0$. Then $\lim_{t\to+\infty}\|\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_{10})-\mathbf{y}(t,\mathbf{x}_0,$

$\mathbf{y}_{20})\| = \lim_{t\to+\infty}\|\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_{10}) - \mathbf{H}(\mathbf{x}(t,\mathbf{x}_{10})) + \mathbf{H}(\mathbf{x}(t,\mathbf{x}_{10})) - \mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_{20})\| \leq \lim_{t\to+\infty}\|\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_{10}) - \mathbf{H}(\mathbf{x}(t,\mathbf{x}_{10}))\| + \lim_{t\to+\infty}\|\mathbf{y}(t,\mathbf{x}_0,\mathbf{y}_{20}) - \mathbf{H}(\mathbf{x}(t,\mathbf{x}_{10}))\| = 0.$

"$\Leftarrow$" (*sufficiency*). Let $\phi_x^t:\mathbb{R}^n\to\mathbb{R}^n$ be the flow of system $D_*^{\bar{\alpha}}\mathbf{x}=\mathbf{f}(\mathbf{x})$ and $\Phi(t)=(\phi_x^t,\phi_y^t)$ be the flow of (1) with $\phi_y^t:\mathbb{R}^{n+m}\to\mathbb{R}^m$. To explicitly construct the map $\mathbf{H}$, choose an arbitrary point $\mathbf{x}_0\in B_x$ and determine the corresponding image point $\mathbf{y}_0=\mathbf{H}(\mathbf{x}_0)$. Since all states $\mathbf{y}\in B_y$ of the response system converge asymptotically to the manifold M, consider the trajectories starting at the past point $(\phi^{-t}(\mathbf{x}_0),\mathbf{y}_0)$. When this trajectory passes the point $\mathbf{x}_0$ the time t has elapsed, and the point $(\mathbf{x}_0,\phi(t,\mathbf{y}_0))$ is closer to M the larger t is. Formally define $\tilde{\mathbf{H}}(\mathbf{x}_0,\mathbf{y}_0)=\lim_{t\to+\infty}\phi_y^t(\phi_x^{-t}(\mathbf{x}_0),\mathbf{y}_0)$. Asymptotical stability implies $\lim_{t\to+\infty}\|\phi_y^t(\phi_x^{-t}(\mathbf{x}_0),\mathbf{y}_{10})-\phi_y^t(\phi_x^{-t}(\mathbf{x}_0),\mathbf{y}_{20})\|=0$ for all $y_{10},y_{20}\in B_y$, and therefore $\tilde{\mathbf{H}}(\mathbf{x}_0,\mathbf{y}_0)$ is independent of $\mathbf{y}_0$. The transformation $\mathbf{H}$ defining the synchronization manifold M is thus given by $\mathbf{H}(\mathbf{x}_0)=\tilde{\mathbf{H}}(\mathbf{x}_0,\mathbf{y}_0)$ for any $\mathbf{y}_0\in B_y$. The proof is complete.

B. Methods to detect generalized synchronization for fractional order systems

According to the analysis of Sec. II A, the most direct numerical method of detecting GS is to compute the CLEs of (1). There exist four representative approaches to compute the LEs of classical systems: Apply the time series to reconstruct the Jacobian matrix of the original system, and get the LEs via the reconstructed matrix [35]; based on Takens's embedding theorem [36] and time series, obtain the LEs [37]; acquire the LEs by using the technique of multivariable feedback network estimation [38]; or just utilizing the delayed coordinates, directly gain the largest LE [39]. The first two approaches depend on the Jacobi matrix of the original system, but the Jacobi matrix does not have a definite meaning for fractional differential systems. However, the third and fourth approaches are still effective for fractional differential systems.

The most powerful (numerical and analytical) method for detecting GS of fractional order systems is the so-called *auxiliary system approach* [24]. First, construct an auxiliary system that is an exact replica of the response system and is driven by a signal from the driving system in the same fashion as the response system, in other words, the response system and auxiliary system have the same vector fields. Now the systems (1) become

$$D_*^{\bar{\alpha}}\mathbf{x} = \mathbf{f}(\mathbf{x}),$$

$$D_*^{\bar{\beta}}\mathbf{y} = \mathbf{g}(\mathbf{y},\mathbf{u}) = \mathbf{g}(\mathbf{y},\mathbf{h}(\mathbf{x})),$$

$$D_*^{\bar{\beta}}\mathbf{z} = \mathbf{g}(\mathbf{z},\mathbf{u}) = \mathbf{g}(\mathbf{z},\mathbf{h}(\mathbf{x})). \quad (4)$$

Because of the *theorem* above, the discussion of GS of (1) is converted to a study of the stability of the manifold $\bar{M}=\{(\mathbf{y},\mathbf{z}):\mathbf{y}=\mathbf{z}\}$ in the subset $\bar{B}=B_y\times B_y\subset\mathbb{R}^m\times\mathbb{R}^m$, i.e., an estimation of whether all trajectories $\mathbf{y}(t)$, $\mathbf{z}(t)$ of (4) with initial conditions in the basin $\bar{B}$ approach $\bar{M}$ as time t tends

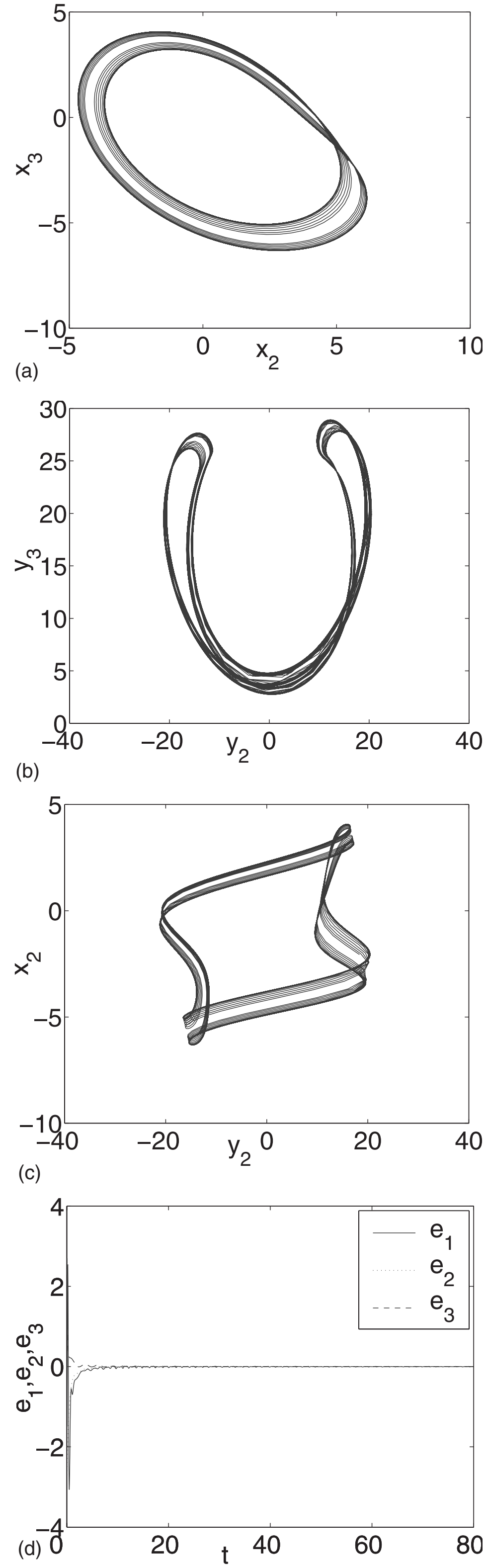

FIG. 1. Generalized synchronization between a fractional order Rössler system (drive) and a fractional order Lorenz system (response) ($\alpha_1=0.82$, $\alpha_2=0.86$, $\alpha_3=0.83$). (a) Fractional order Rössler system x_2 vs x_3 ($a_1=1.6$, $b_1=0.68$, $c_1=5.3$). (b) Fractional order Lorenz system y_2 vs y_3 ($\sigma=10$, $r=28$, $b_2=2.666$). (c) Fractional order Rössler system x_2 vs fractional order Lorenz system y_2. (d) $e_1=y_1-z_1$, $e_2=y_2-z_2$, and $e_3=y_3-z_3$.

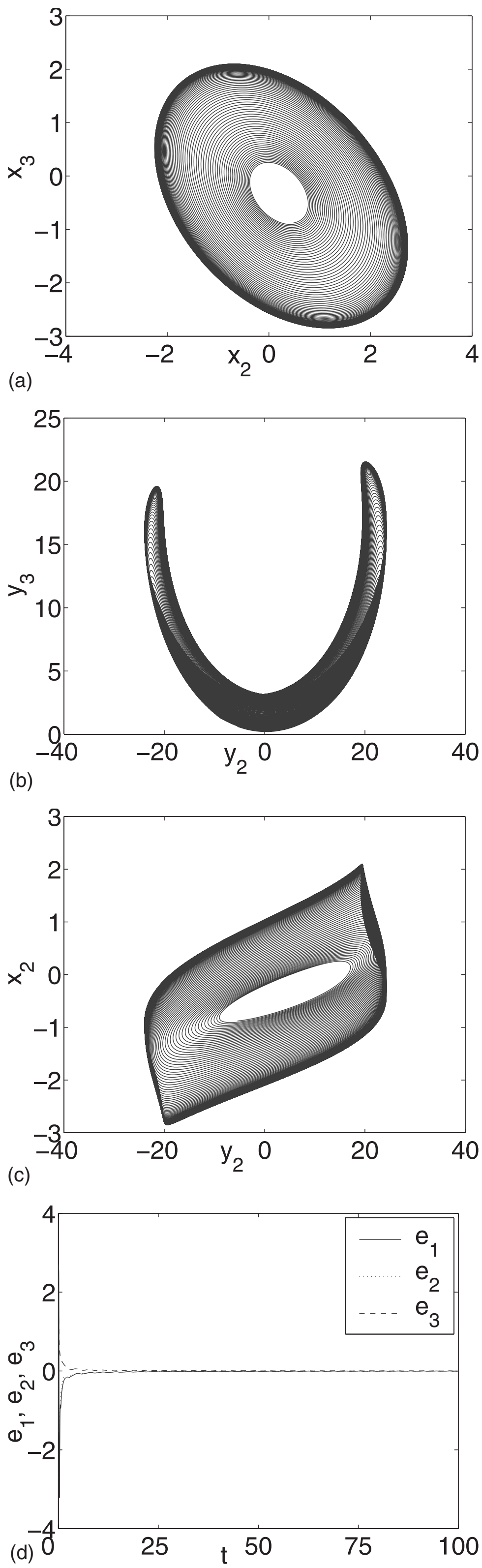

FIG. 2. Generalized synchronization between a fractional order Rössler system ($\alpha_1=0.80, \alpha_2=0.80, \alpha_3=0.80$) and a fractional order Lorenz system ($\beta_1=0.69, \beta_2=0.66, \beta_3=0.83$). (a) Fractional order Rössler system x_2 vs x_3 ($a_1=1.6, b_1=0.68, c_1=5.3$). (b) Fractional order Lorenz system y_2 vs y_3 ($\sigma=10, r=28, b_2=2.666$). (c) Fractional order Rössler system x_2 vs fractional order Lorenz system y_2. (d) $e_1=y_1-z_1$, $e_2=y_2-z_2$, and $e_3=y_3-z_3$.

to infinity. Now studying the stability of the manifold M, which in general has a complicated shape, is transformed to studying the stability of the simple manifold $\bar{M}$. In numerical simulation and laboratory experiments, the stability of the manifold of identical oscillations can be easily verified by means of observation of the regime of stable identical oscillations of the response and auxiliary systems, namely, the response is stable if two copies of a driven system show the same response.

The third method is based upon the statistical test called *mutual false nearest neighbors* to determine when closeness in response space implies closeness in driving space. Since this method does not depend on the concrete form of the dynamical system but just the generating time series, it also works well for fractional differential equations (for more details the reader can refer to Ref. [26]).

III. THREE KINDS OF GENERALIZED SYNCHRONIZATION FOR FRACTIONAL ORDER SYSTEMS

Because of the nonlocal property of fractional derivatives [40], it is always difficult to efficiently do numerical computations for fractional order systems in the time domain for a long time. Luckily, the present author in Ref. [41] successfully circumvented this difficulty by further exploring the *short memory principle* of fractional derivatives. That is to say, the computational cost is reduced from $O(n^2)$ to $O(n \log n)$, where n is the number of points used in computation. In this section, we numerically investigate three kinds of GS for fractional order systems by using three examples, with the numerical scheme mentioned in Ref. [42]. Furthermore, the three examples illustrate that GS and equivalence of fractional order systems are related but independent notions.

The first example shows the GS for arbitrary pairs of fractional order systems with the same fractional orders. The divergence and convergence rates of fractional order systems are algebraic ($t^{\pm\alpha}$, where α is the order of the fractional order system) rather than exponential (as are those of classical systems). Taking the same fractional orders for the corresponding equations of the drive and response systems implies that the two systems have the same convergence rate. Here a fractional order Lorenz system [43,44] is driven by a fractional order Rössler system [45]. The drive system is given by

$$D_*^{\alpha_1} x_1 = a_1 + x_1(x_2 - c_1),$$

$$D_*^{\alpha_2} x_2 = - x_1 - x_3,$$

$$D_*^{\alpha_3} x_3 = x_2 + b_1 x_3, \qquad (5)$$

the equations of the response system are

$$D_*^{\alpha_1} y_1 = - \sigma(y_1 - y_2),$$

$$D_*^{\alpha_2} y_2 = r u(t) - y_2 - u(t) y_3,$$

$$D_*^{\alpha_3} y_3 = u(t) y_2 - b_2 y_3, \tag{6}$$

and the auxiliary system is

$$D_*^{\alpha_1} z_1 = -\sigma(z_1 - z_2),$$

$$D_*^{\alpha_2} z_2 = r u(t) - z_2 - u(t) z_3,$$

$$D_*^{\alpha_3} z_3 = u(t) z_2 - b_2 z_3,$$

where $u(t)$ is a scalar function of x_1, x_2, x_3. Figure 1 shows attractors from (5) and (6) for the case $u(t)=x_1+x_2+x_3$. Because of the GS the attractor of the $\mathbf{y}$ system (6) shown in Fig. 1(b) is a nonlinear image of the attractor of the $\mathbf{x}$ system (5) given in Fig. 1(a). The $\mathbf{x}_2$ vs $\mathbf{y}_2$ diagram in Fig. 1(c) shows that the two systems are not synchronized in the sense of CS. Figure 1(d) displays $\mathbf{e}=\mathbf{y}-\mathbf{z}$, which illustrates the occurrence of GS by the auxiliary system approach.

For the second example we still take a fractional order Rössler system as the drive system and a fractional order Lorenz system as the response system but with different orders. So the divergence and convergence rates of the drive and response systems are different but GS occurs. In fact, the orders of fractional order systems are also important parameters of real models and it is difficult to get two fractional order systems with completely the same orders in practical applications. Similar to the first example, the drive, response, and auxiliary systems are described as follows:

$$D_*^{\alpha_1} x_1 = a_1 + x_1(x_2 - c_1),$$

$$D_*^{\alpha_2} x_2 = -x_1 - x_3,$$

$$D_*^{\alpha_3} x_3 = x_2 + b_1 x_3, \tag{7}$$

$$D_*^{\beta_1} y_1 = -\sigma(y_1 - y_2),$$

$$D_*^{\beta_2} y_2 = r u(t) - y_2 - u(t) y_3,$$

$$D_*^{\beta_3} y_3 = u(t) y_2 - b_2 y_3, \tag{8}$$

and

$$D_*^{\beta_1} z_1 = -\sigma(z_1 - z_2),$$

$$D_*^{\beta_2} z_2 = r u(t) - z_2 - u(t) z_3,$$

$$D_*^{\beta_3} z_3 = u(t) z_2 - b_2 z_3.$$

The attractors from (7) and (8) for the case $u(t)=x_1+x_2+x_3$ are displayed in Fig. 2.

The last example shows the GS for a classical system and its corresponding fractional order system with mismatched parameters. These two systems are obviously not equivalent since the classical derivative is a local operator and the fractional derivative is a nonlocal one. Moreover the divergence and convergence rates of fractional order systems are algebraic ($t^{\pm\alpha}$, where α is the order of the fractional order system), but those of classical systems are exponential. With the

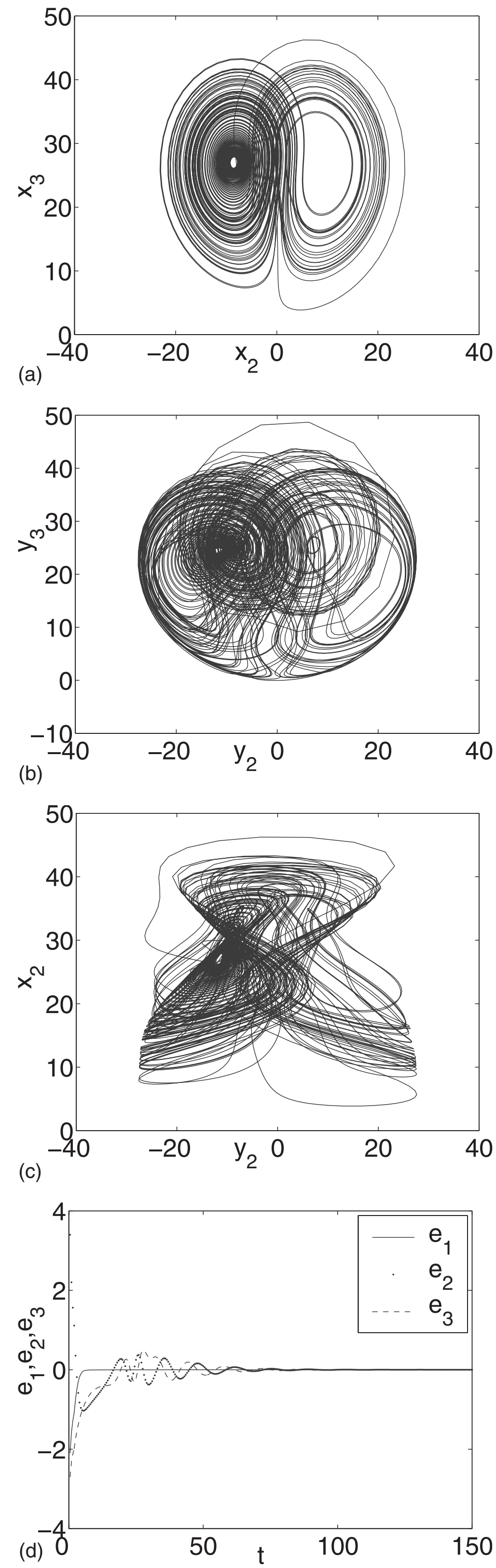

FIG. 3. Generalized synchronization between a classical Lorenz system and a fractional order Lorenz system ($\beta_1=0.80, \beta_2=0.80, \beta_3=0.80$). (a) Classical Lorenz system x_2 vs x_3 ($\sigma_1=8, r_2=28, b_2=2.666$). (b) Fractional order Lorenz system y_2 vs y_3 ($\sigma_2=10, r_2=28, b_2=2.666$). (c) Classical Lorenz system x_2 vs fractional order Lorenz system y_2. (d) $e_1=y_1-z_1$, $e_2=y_2-z_2$, and $e_3=y_3-z_3$.

classical Lorenz system as the drive system and the fractional order Lorenz system as the response system, the equations of the drive system are

$$dx_1/dt = -\sigma_1 x_1 + \sigma_1 x_2,$$

$$dx_2/dt = r_1 x_1 - x_2 - x_1 x_3,$$

$$dx_3/dt = x_1 x_2 - b_1 x_3, \tag{9}$$

the response system is presented as

$$D_*^{\beta_1} y_1 = -\sigma_2 y_1 + u(t),$$

$$D_*^{\beta_2} y_2 = r_2 y_1 - y_2 - y_1 y_3,$$

$$D_*^{\beta_3} y_3 = y_1 y_2 - b_2 y_3, \tag{10}$$

and the auxiliary system is

$$D_*^{\beta_1} z_1 = -\sigma_2 z_1 + u(t),$$

$$D_*^{\beta_2} z_2 = r_2 z_1 - z_2 - z_1 z_3,$$

$$D_*^{\beta_3} z_3 = z_1 z_2 - b_2 z_3,$$

where $u(t) = \sigma_1 x_2$. Figure 3 demonstrates attractors from (9) and (10).

IV. CONCLUSIONS

Synchronization has a lot of potential applications, but in practice, it is difficult to find two completely identical systems even when we try to construct them. So the synchronization of nonlinear systems that are not (completely) identical is critical for the appearance of synchronized motions in realistic systems where precise identity of the systems is unlikely. The present paper reveals many interesting forms of GS of fractional order systems, including synchronization between two different fractional order systems with the same orders, synchronization between two different fractional order systems with different orders, and synchronization between a classical system and its corresponding fractional order system with mismatched parameters. A general criterion for the occurrence of GS of unidirectionally fractional order coupled systems is presented. Three methods to detect GS of fractional order systems are discussed, and in particular the auxiliary system approach is used to detect the GS of three canonical examples. But it must be noted that, if there are multiple basins of attraction for the response system, then the auxiliary system approach could fail. In this case we can turn to the first and second methods mentioned in Sec. II. Finally, we also want to emphasize that for the GS of bidirectional coupled systems, general criteria, except for negative CLEs, are difficult to obtain not only for fractional order systems but also for classical systems.

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